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EVOLUTION OF HOMOGENEOUS TURBULENCE IN A DENSITY-STRATIFIED MEDIUM. 1. ANALYSIS IN FAR FIELD

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The equations presented in [1] for the evolution of homogeneous turbulence in a density-stratified medium are investigated analytically. The problem involves the small parameter $\varepsilon = Fr^2$. Applying variants of the small-parameter method, it is possible to calculate the frequency of internal waves and a number of asymptotic regimes in the development of turbulent velocity and scalar fields over time. The so-called far field of evolution, i.e., the field of large values of τ ($\tau \gg 1$), is considered, in which the differential order of the initial system of equations is lowered.

In a number of recent experimental and theoretical works devoted to analysis of the decay of free turbulence in a density-stratified medium, periodic changes in the turbulent mass flux, square of velocity fluctuations, square of scalar fluctuations, as well as in other characteristics of turbulence have been discovered. A rather complete analytical review of these publications can be found in [1]. We can mention additionally work [2], in which, using a direct numerical simulation on the basis of Navier–Stokes equations and statistical averaging over an ensemble, oscillations of statistical moments were discovered and several periods from the start of the process were calculated. Application of the method of direct numerical simulation is limited to molecular Prandtl numbers close to unity and to small time intervals. Experimental methods also do not permit investigation of turbulence decay over large time intervals, in view of the smallness of the quantities measured.

In practically important cases of the atmosphere and ocean, the theory of homogeneous turbulence can be an adequate alternative method for investigating turbulent fields over very large periods of time. This cannot be done by other investigation methods.

It is known that the evolution of turbulence in a density-stratified medium is essentially a two-scale process, in which disturbances with large spatial scales can be attributed to buoyancy forces, and those with fine scales, to viscous dissipation [3]. The time scales corresponding to these two processes also differ, and this is used in the present work.

We consider the evolution of turbulence produced by a turbulizing grid in a medium with a homogeneous velocity field in the presence of a constant transverse density gradient caused by the gravity force field.

The initial system of equations in the present work coincides with that of [1], which was obtained by a more complex second-order model developed in [4]. Analytical investigation of the equations of homogeneous turbulence that contain the small parameter ε will aid in explaining the trends discovered earlier, describing the dependence of the solution on the Prandtl and Froude numbers, and determining the dependence of the frequency and amplitude of vibrations on the factors indicated, and, on this basis, will allow conclusions to be drawn about the velocity and mechanism of turbulence decay in a density-stratified medium.

The system of equations from [1] can be written in dimensionless form as

$$\frac{dR_{22}}{d\tau} = -2\left\{\frac{1}{3}\left[d + \frac{9}{2}(1-d)\right]\left(3\frac{R_{22}}{E} - 1\right) + \frac{1}{3} + \frac{4}{5}Q\frac{T_u}{E}\mathrm{Fr}^2\right\}\frac{E}{T_u},$$

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$$\frac{dE}{d\tau} = -2\left[1 + Q\frac{T_u}{E}\mathrm{Fr}^2\right]\frac{E}{T_u},$$

$$\frac{dT_u}{d\tau} = (F_u^{**} - 2) - 2\left[1 - d\left(\frac{2\sigma}{1+\sigma}\right)\left(\sigma_\infty + \frac{3}{5}\right)\frac{1}{R_\infty}\frac{T_u}{T_\rho}\right]\mathrm{Fr}^2\frac{QT_u}{E},$$

$$\frac{dQ}{d\tau} = -\mathrm{Fr}^2\left[\frac{2}{3} - \frac{1}{\mathrm{Fr}^2}\frac{R_{22}}{\Theta} - d\left(\frac{R_{22}}{E} - \frac{1}{3}\right)\right]\Theta -$$

$$-\left[(1-d)\left(\frac{1}{3} + 10\frac{R_{22}T_u}{ET_\rho}\right) + 2d\left(\sigma_\infty + \frac{3}{5}\right)\frac{1}{R_\infty}\frac{T_u}{T_\rho}\right]\frac{Q}{T_u} \tag{1}$$

$$\frac{d\Theta}{d\tau} = -2\left[1 - Q\frac{T_\rho}{\Theta}\right]\frac{\Theta}{T_\rho}$$

$$\frac{dT_\rho}{d\tau} = (F_{\rho 2}^{**} - 2) + F_{\rho 1}^{**}\frac{T_\rho}{T_u} - d\frac{4}{3}\left(1 - \frac{3}{5R_\infty}\right),$$

where the functions of interaction of turbulent vortices of different scales for the velocity and scalar fields, F_u^{**}, $F_{\rho 1}^{**}$, $F_{\rho 2}^{**}$, are calculated on the basis of expressions given in [1]:

$$F_u^{**} = \frac{11}{3} - \frac{13}{15}d, \quad F_{\rho 1}^{**} = \frac{5}{3}(1-d), \quad F_{\rho 2}^{**} = 2 + \frac{4}{3}d,$$

$d(R_\lambda^2) = 1 - 2\sqrt{1 + \delta_u/R_\lambda^2}$ is the parameter of the interaction of turbulent vortices of different scales from [4], $d \in (0.1)$; $\delta_u \approx 2800$, σ_∞ is the asymptotic value of the turbulent Prandtl number for $\tau \to \infty$, and $\mathrm{Fr} = 0$, R_∞ is the asymptotic value of the ratio of scales T_u/T_ρ for $\tau \to \infty$ and $\mathrm{Fr} = 0$. The asymptotic values of σ_∞ and R_∞ are taken from [5]:

$$\sigma_\infty = \frac{3(1-\sigma)}{10\sigma}\left[1 - \left(\frac{2\sigma}{1+\sigma}\right)^{3/2}\right]^{-1}, \tag{2}$$

$$R_\infty = \frac{1}{5\sigma}\left[1 - \left(\frac{2\sigma}{1+\sigma}\right)^{3/2} + \sigma^{3/2}\right]\left[1 - 2\left(\frac{2\sigma}{1+\sigma}\right)^{1/2} + \sigma^{1/2}\right]^{-1}. \tag{3}$$

Note that the Froude number introduced in [1] represents the ratio of the mass lift forces to the forces of inertia and actually is the inverse of the generally accepted one. For the majority of practically important cases this parameter is very small. The experimental data published in the literature relate either to atmospheric air or to salt water. Estimation of the parameter $\varepsilon = \mathrm{Fr}^2$ entering into Eq. (1) for these most important media (if we exclude very saturated saline solutions from consideration) shows that it is of the order of 10^{-3}–10^{-4}.

The system of differential equations with the small parameter ε can be effectively solved analytically by approximate methods of expansion in terms of the small parameter, and frequently the quality of such a solution, i.e., its simplicity and applicability to the analysis, exceeds the quality of a numerical solution. On the other hand, such a solution is nevertheless formal and must be confirmed to a sufficient extent by other methods of investigation.

To use the small-parameter method, it is necessary to expand all the functions in Eqs. (1) in powers of ε and unite the terms with the same power of ε into solvable subsystems. A direct expansion in the small parameter ε in system (1) turns out to be singular for the region with $\tau > 1$ in the sense of asymptotic expansions [6], since such a solution will involve positive powers of τ that grow with the number of approximation. As shown below, such a singularity is caused by the decrease in the functional order of the system at large values of τ. To obtain an asymptotic expansion in the small parameter that is applicable at large values of τ, it is necessary to perform a regularizing replacement of variables and coordinates by introducing the small parameter ε into them.

The a priori information on the solution suggests the class of functions from which one should construct its approximate analog. According to the results of [1], the behavior of the functions of the system can be presented in the form of monotonic relations, on which decaying vibrations are imposed, with the amplitude and frequency of oscillations varying very slowly at large values of τ. Proceeding from the form of numerical solution, an approximate solution for all the functions f in Eqs. (1) can be constructed as a sum of a harmonic function of variable amplitude $\tilde{f}$ and of a certain smooth component $\hat{f}$, i.e., $f = \hat{f} + \tilde{f} = \hat{f} + \tilde{f}'\tilde{f}''$, where $\tilde{f}'$ is the amplitude and $\tilde{f}''$ is the harmonic function, with these functions being dependent on different time scales.

According to this representation of the unknown functions, we will construct an expansion in the small parameter whose terms, being grouped by identical powers of ε, will give systems of differential equations for determining $\hat{f}$, $\tilde{f}'$, and $\tilde{f}''$. When possible, these systems will be solved analytically, but even when it is impossible, such a division into $\hat{f}$, $\tilde{f}$, and $\tilde{f}''$ turns out to be very useful, since it allows one to investigate the frequency of vibrations and the rates of degeneration of various functions.

We introduce the following set of new variables and coordinates:

$$q = \varepsilon T_p Q/E\,,\quad \vartheta = \varepsilon\Theta/E\,,\quad K = R_{22}/E\,,\quad R = T_u/T_p\,,\quad t = \varepsilon T_p\,,\quad \tilde{\tau} = e\tau\,, \tag{4}$$

where $e = \varepsilon^{1/2}$. The function K represents the fraction of the energy of transverse pulsations in the kinetic energy of turbulence. The function R is the ratio of the time scales of the velocity and scalar fields, the function ϑ is the ratio of the potential energy of the scalar field to the kinetic energy of turbulence, and the functions t and $\tilde{\tau}$ are used below as independent variables. In this notation, the initial system of equations (1) can be written in a shorter form which is more suitable for analysis:

$$t\frac{dK}{d\tau} = \varepsilon\frac{-7d'(K - 1/3)}{R} + 2\varepsilon q(K - 4/5)\,,$$

$$t\frac{dR}{d\tau} = \varepsilon\frac{4}{5}d(1 - R/R_\infty) - 2\varepsilon q(1 - \alpha_2 R)R\,,$$

$$t\frac{d\vartheta}{d\tau} = 2\varepsilon\left(\frac{1}{R} - 1\right)\vartheta + 2\varepsilon q(1 + \vartheta)\,, \tag{5}$$

$$t\frac{dq}{d\tau} = t^2 A_1 + \varepsilon q\left[\frac{1}{R}\left(2 - \frac{d'}{3}\right) - 10d'K - \alpha_1 + p\right] + 2\varepsilon q^2\,,$$

$$\frac{dt}{d\tau} = \varepsilon p\,,$$

$$t\frac{dE}{d\tau} = -\frac{2\varepsilon E}{R} - 2\varepsilon qE\,,$$

where

$$d' = 1 - d\,,\quad \alpha_1 = 2d\left(\sigma_\infty + \frac{3}{5}\right)/R_\infty\,,\quad \alpha_2 = d\left(\frac{2\sigma}{1+\sigma}\right)\left(\sigma_\infty + \frac{3}{5}\right)/R_\infty\,,$$

$$p = \frac{4d}{5R_\infty} + \frac{5d'}{3R} > 0 \quad\text{and}\quad A_1 = K + \vartheta\left[d\left(K - \frac{1}{3}\right) - \frac{2}{3}\right]. \tag{6}$$

In the autonomous differential system (5) it is possible to introduce t instead of r as a new independent variable. Then it becomes evident that for $\varepsilon \to 0$ the fourth equation in (5) degenerates into an algebraic one:

$t^2 A_1 = 0$. This degenerate case corresponds to high values of t in Eqs. (5), since in this case the term $t^2 A_1$ becomes predominant in the fourth equation. A solution corresponding to such degeneration is called external, and in a certain sense it should be akin to the solution of the initial problem (5).

We note that at d = const the equation for E in system (5) can be solved separately from the system, since in this case the first five equations do not contain a dependence on E. Such a situation is observed in the case of asymptotically large ($R_\lambda \gg 1$) and asymptotically small ($R_\lambda \ll 1$) values of turbulent Reynolds numbers. At turbulent Reynolds numbers not satisfying these two extreme cases, it is more convenient for the analysis to replace the equation for E in system (5) by the equation for d. This is equivalent to the equation for the turbulent Reynolds number. The differential equation for the function d can easily be obtained from the definition of this function and from system (5):

$$t \frac{d(d)}{d\tau} = \varepsilon p_1 \left[\frac{F_u^{**} - 4}{R} - 2(2 - \alpha_2 R) q \right], \tag{7}$$

where p_1 is the logarithmic derivative of d with respect to R_λ^2. Its computation yields

$$p_1 = \frac{d(d)}{d(R_\lambda^2)} R_\lambda^2 = (1/(1 + \delta_u / R_\lambda^2)^{1/2} - 1)(1 + (1 + \delta_u / R_\lambda^2)^{1/2})^{-1} = -\frac{dd'}{1 + d}.$$

The function $t(\tau)$ in system (5) increases monotonically with time and can be used as an independent variable that replaces τ in Eqs. (5) and (7). We shall use the independent variable t not instead of τ but together with τ (more precisely, with $\tilde{\tau}$) applying a variant of the well-known method of many scales in which expansion of derivatives is made in several variables [6]. Thus, let us represent the unknown functions in the form of expansions:

$$K = \hat{K}(t) + \varepsilon \tilde{K}(t, \tilde{\tau}) + O(\varepsilon^2), \quad R = \hat{R}(t) + \varepsilon \tilde{R}(t, \tilde{\tau}) + O(\varepsilon^2), \quad \vartheta = \hat{\vartheta}(t) + \varepsilon \tilde{\vartheta}(t, \tilde{\tau}) + O(\varepsilon^2),$$

$$d = \hat{d}(t) + \varepsilon \tilde{d}(t, \tilde{\tau}) + O(\varepsilon^2), \quad q = \hat{q}(t) + e \tilde{q}(t, \tilde{\tau}) + O(e^2), \tag{8}$$

where $O(\ldots)$ are functions of the order. It is shown below that the first terms correspond to the smooth component of the solution onto which the vibrations in the form of the second term are imposed in Eq. (8). We note that the splitting up of all the functions, except q, is made in ε and of q in $e = \varepsilon^{1/2}$. Such an approach makes it possible to simplify representation, since then it is not necessary to apply an asymptotic-union procedure. On the other hand, the alternative method of matched asymptotic expansions (MAE) allows one to substantiate precisely this structure of the assumed approximate solution, i.e., the presence of various degrees in the expansions for q and other functions. We will consider in brief the scheme of the method of MAE for the given problem.

According to this method, a solution can be constructed in the form of a sum of internal and external solutions, minus their coinciding part [6]. In this case, the external solution is represented by circumflexed functions. The internal expansion is carried out using the "stretched" variable $\tilde{\tau}$. For a first approximation of the internal expansion in e a simple solution is obtained: q is a harmonic function, and all the remaining functions are constants, which, as ascertained from the principle of asymptotic-union, are equal to the external solutions for the corresponding functions. Substituting these solutions into the system of the second approximation of the internal expansion, we find the oscillations of the remaining functions; they will have the factor $e^2 = \varepsilon$.

Let us substitute expansion (8) into system (5) and consider, for example, the first equation in system (5). Substituting into it Eq. (8), we obtain the following equation accurate to terms of the order of $e\varepsilon$ inclusively:

$$\varepsilon p t \frac{d\hat{K}}{dt} + e \varepsilon t \frac{\partial \tilde{K}}{\partial \tilde{\tau}} = -7\varepsilon \frac{\hat{d}'(\hat{K} - 1/3)}{\hat{R}} + 2\varepsilon \hat{q}(\hat{K} - 4/5) + 2e\varepsilon \tilde{q}(\hat{K} - 4/5).$$

Collecting in this equation terms of different orders of smallness and assuming from the definition of the far region that $\tau \gg \varepsilon$, we obtain

$$pt\frac{d\hat{K}}{dt} = -7\frac{\hat{d}'(\hat{K}-1/3)}{\hat{R}} + 2\hat{q}(\hat{K}-4/5),\quad t\frac{\partial\tilde{K}}{\partial\tilde{\tau}} = 2\tilde{q}(\hat{K}-4/5).$$

Proceeding in a similar manner in the other equations of system (5), we write two systems. The system of equations for the circumflexed functions has the form:

$$\begin{aligned}
&pt\frac{d\hat{K}}{dt} = -7\frac{\hat{d}'(\hat{K}-1/3)}{\hat{R}} + 2\hat{q}(\hat{K}-4/5),\\
&tp\frac{d\hat{R}}{dt} = \frac{4}{5}\hat{d}(1-R/\hat{R}_\infty) - 2(1-\alpha_2\hat{R})\hat{R}\hat{q},\\
&t^2\hat{A}_1 = 0,\\
&tp\frac{d\hat{\vartheta}}{dt} = 2\left(\frac{1}{\hat{R}}-1\right)\hat{\vartheta} + 2\hat{q}(1+\hat{\vartheta}),\\
&tp\frac{d(\hat{d})}{dt} = p_1\left[\frac{F_u^{**}-4}{\hat{R}} - 2(2-\alpha_2\hat{R})\hat{q}\right],\\
&tp\frac{d\hat{E}}{dt} = -\frac{2\hat{E}}{\hat{R}} - 2\hat{q}\hat{E},
\end{aligned}\tag{9}$$

where it is assumed that the functions α_1, α_2, $\hat{A}_1$, P_1, and F_u^{**} depend on the circumflexed functions, i.e., on the smooth components in expansions (8).

The oscillations of the functions in system (5) are described by the following system:

$$\begin{aligned}
&t\frac{\partial\tilde{K}}{\partial\tilde{\tau}} = 2\tilde{q}(\hat{K}-4/5),\quad t\frac{\partial\tilde{R}}{\partial\tilde{\tau}} = -2\tilde{q}(1-\alpha_2\hat{R})\hat{R},\quad t\frac{\partial\tilde{q}}{\partial\tilde{\tau}} = t^2\tilde{A}_1 + f(t),\\
&t\frac{\partial\tilde{\vartheta}}{\partial\tilde{\tau}} = 2\tilde{q}(1+\hat{\vartheta}),\quad t\frac{\partial(\tilde{d})}{\partial\tilde{\tau}} = -2p_1\tilde{q}(2-\alpha_2\hat{R}),\quad t\frac{\partial\tilde{E}}{\partial\tilde{\tau}} = -2\tilde{q}\hat{E},
\end{aligned}\tag{10}$$

where

$$f(t) = -t\frac{d\hat{q}}{dt} + \hat{q}\left[\hat{R}^{-1}\left(2-\frac{\hat{d}'}{3}\right) - 10\hat{d}'\hat{K} - \alpha_1 + p\right] + 2\hat{q}^2,\tag{11}$$

and $\tilde{A}_1$ is represented by the second term of the expansion for the function A_1

$$A_1 = \hat{A}_1(\hat{K},\hat{\vartheta},\hat{d}) + \varepsilon\tilde{A}_1.\tag{12}$$

The equation for the amplitude of oscillations of q follows from the terms of expansions with the power $e\varepsilon$ in the fourth line of system (5):

$$tp\frac{\partial\tilde{q}}{\partial t} = \tilde{q}\left[\frac{1}{\hat{R}}\left(2-\frac{\hat{d}'}{3}\right) - 10\hat{d}'\hat{K} - \alpha_1 + p\right] + 4\tilde{q}\hat{q},\tag{13}$$

As shown below, the amplitudes of oscillations of the remaining functions are proportional to the amplitude of vibrations of the mass flow.

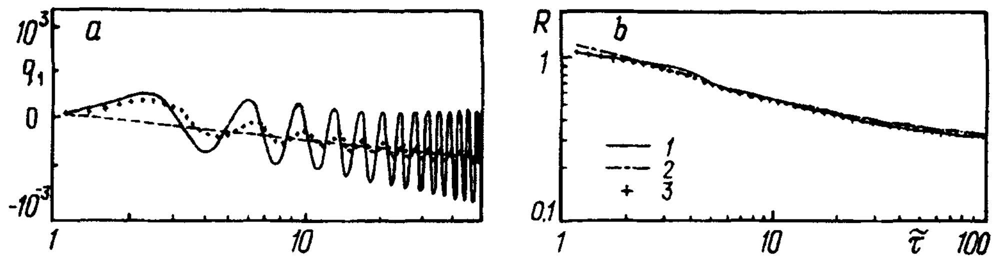

Fig. 1. Dependence of the turbulent mass flow $q_1 = T_\rho Q/E$ and of the ratio of time scales $R = T_u/T_\rho$ on time $\tilde{\tau} = \mathrm{Fr}\tau$; 1) numerical solution of system of equations (5)-(7); 2) numerical solution of system of equations (9) and (15); 3) data of [1]; a) $q_1(\tilde{\tau})$; b) $R(\tilde{\tau})$.

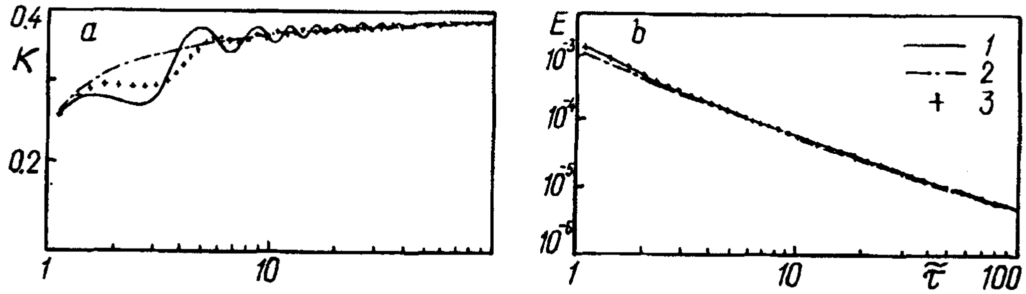

Fig. 2. Dependence of the ratio $K = R_{22}/E$ and of turbulent kinetic energy E on time $\tilde{\tau} = \mathrm{Fr}\tau$; 1) numerical solution of system (5)-(7); 2) numerical solution of system (9) and (15); 3) calculated data of [1]; a) $K(\tilde{\tau})$; b) $E(\tilde{\tau})$.

The system of equations (9) for the circumflexed functions coincides with the degenerate system for $\varepsilon \to 0$ in system (5). The meaning of the degenerate system follows from expansions (8), i.e., it is a system that describes the behavior of the functions averaged over oscillations. The value of the variable t in Eqs. (9) is not small; consequently, $\hat{A}_1 = 0$.

Differentiating this relation with respect to t, we obtain

$$pt \frac{d\hat{A}_1}{dt} = 2\varepsilon\,(c_1 + c_2\hat{q}) = 0\,, \tag{14}$$

where

$$c_1 = -\frac{7}{2}(1 + \hat{\vartheta}\hat{d})\,\hat{d}'\,(\hat{K} - 1/3)\,\hat{R}^{-1} + \hat{\vartheta}\,[\hat{d}\,(K - 1/3) - 2/3]\left(\frac{1}{\hat{R}} - 1\right) + \hat{\vartheta}\,(\hat{K} - 1/3)\,\frac{p_1}{2}\,(F_u^{**} - 4)\,\frac{1}{\hat{R}}\,,$$

$$c_2 = (1 + \hat{\vartheta}\hat{d})\,(\hat{K} - 4/5) + [\hat{d}\,(\hat{K} - 1/3) - 2/3]\,(1 + \hat{\vartheta}) + \hat{\vartheta}\,(\hat{K} - 1/3)\,p_1\,(2 - \alpha_2\hat{R})\,.$$

Let us express $\hat{q}$ from Eq. (14) as

$$\hat{q} = -\,c_1/c_2\,. \tag{15}$$

Relation (15) in the degenerate system for smooth components replaces the fourth line of system (5). In this case, the expression $\hat{A}_1 = 0$ itself is used to find $\hat{\vartheta}$, and thus it replaces the third line in system (5) in the system of equations for the circumflexed functions. Otherwise, this system of equations (we will denote it as (9) and (15)) coincides with system (5)-(7).

As follows from comparison of Figs. 1 and 2, the solution of the system of equations (9) and (15) corresponds well on the average to the oscillating numerical solution of system (5)-(7). Just as in [1], the initial conditions were prescribed from the experimental values of [7] for water (σ = 800, Fr = $3.67 \cdot 10^{-2}$, M =

$5.08 \cdot 10^{-2}$m) and corresponded to the following set of data: $R_{22_0} = 3.47 \cdot 10^{-2}$, $E_0 = 1.33 \cdot 10^{-1}$, $T_{\rho_0} = 38.4$, $T_{u_0} = 43.3$, $Q_0 = 3.31 \cdot 10^{-3}$, $\Theta_0 = 0.153$. The amplitude of the oscillations of the function q_1 increases with time, but more slowly than T_ρ, as a result of which the amplitude of the oscillations of the function Q/E decreases with an increase in τ. The oscillations of the functions R and E are almost imperceptible against the background of their mean values, and the oscillations of the ratio K also become small with time compared to $\hat{K}$. The points in Figs. 1 and 2 show the data of [1] for comparison. It is evident that for all the functions there is a certain systematic discrepancy between the present calculation and the data of [1]. As is seen, this unexplained difference mainly concerns the amplitudes of oscillations; the mean values coincide satisfactorily. In what follows it will be shown that in our numerical calculations the amplitudes of oscillations agree excellently with those predicted analytically.

Let us go over to the solution of the system of equations (10). We will perform partial differentiation of the third equation in (10) with respect to $\hat{\tau}$:

$$\frac{\partial^2 \tilde{q}}{\partial \tilde{\tau}^2} = t \frac{\partial \tilde{A}_1}{\partial \tilde{\tau}} . \tag{16}$$

A differential equation for the function A_1 can be obtained by differentiation with respect to τ in Eq. (6). It has the form

$$t \frac{dA_1}{d\tau} = 2\varepsilon (c_1 + c_2 q) , \tag{17}$$

Expanding in this equation in powers of e, according to Eq. (8), we obtain the following equation in addition to Eq. (14):

$$t \frac{\partial \tilde{A}_1}{\partial \tilde{\tau}} = 2c_2 \tilde{q} , \tag{18}$$

Analysis of c_2 shows that this quantity is negative. Therefore, using the notation

$$\omega^2 = -2c_2 , \tag{19}$$

we write for $\tilde{q}$

$$\frac{\partial^2 \tilde{q}}{\partial \tilde{\tau}^2} + \omega^2 \tilde{q} = 0 . \tag{20}$$

The solution of Eq. (20) is a harmonic function of general form multiplied by an arbitrary function of t. Separating variables in Eq. (20) and substituting $\tilde{q} = \tilde{q}'(t)\tilde{q}''(\tilde{\tau})$, we obtain

$$q'' = \sin(\omega\tilde{\tau} + \varphi_0) .$$

Substituting $\tilde{q}$: $\tilde{q} = \tilde{q}'(t) \sin(\omega\tilde{\tau} + \varphi_0)$ into system of equations (11), we write the solution for the wave components of the remaining functions. For example, for the function K we obtain

$$\tilde{K} = \tilde{K}'\tilde{K}'' = -2 \frac{\tilde{q}'(\hat{K} - 4/5)}{t\omega} \cos(\omega\tilde{\tau} + \varphi_0) ,$$

from which we conclude that

$$\tilde{K}' = -2 \frac{\tilde{q}'(\hat{K} - 4/5)}{t\omega} , \quad \tilde{K}'' = \cos(\omega\tilde{\tau} + \varphi_0) . \tag{21}$$

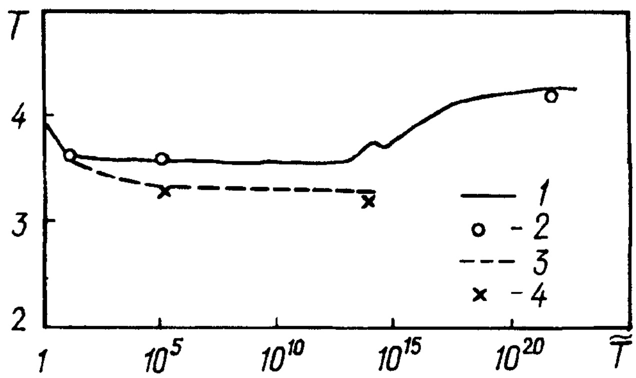

Fig. 3. Comparison of oscillation periods calculated by formula (19) (curves) and those given in [1] (points): 1, 2) for water; 3, 4) for air.

Similarly we determine the amplitudes and phases of the remaining functions:

$$\tilde{R}' = 2\frac{\tilde{q}'(1-\alpha_2\hat{R})\hat{R}}{t\omega}, \quad \tilde{R}^{\bullet} = \cos(\omega\tilde{\tau} + \varphi_0), \tag{22}$$

$$\tilde{\vartheta}' = -2\frac{\tilde{q}'(1+\hat{\vartheta})}{t\omega}, \quad \tilde{\vartheta}^{\bullet} = \cos(\omega\tilde{\tau} + \varphi_0), \tag{23}$$

$$\tilde{d}' = 2p_1\frac{\tilde{q}'(2-\alpha_2\hat{R})}{t\omega}, \quad \tilde{d}^{\bullet} = \cos(\omega\tilde{\tau} + \varphi_0), \tag{24}$$

$$\tilde{E}' = 2\frac{\tilde{q}'\hat{E}}{t\omega}, \quad \tilde{E}^{\bullet} = \cos(\omega\tilde{\tau} + \varphi_0). \tag{25}$$

From Eqs. (21)-(25) it follows that the amplitude of the oscillations of all the functions is proportional to the amplitude of the oscillations of the mass flow; all the functions oscillate with the same period, and the phase of the oscillations of all the functions is shifted by exactly a quarter period from the phase of the oscillations of q. At large values of t, the amplitudes of the oscillations of all the functions, except q, are small. Exactly the same character of change of the functions was noted in the numerical computation in [1]. The relationship of the amplitudes of other functions with the amplitude of the oscillations of q is dictated by relations (21)-(25).

Comparison of the oscillation period $T = 2\pi/\omega$ calculated by Eq. (19) shows complete agreement with [1] within the limits of comparison accuracy (Fig. 3). It can be seen that the periods that differ for $\tau \to \infty$ correspond to cases of water with $\sigma = 800$ and air with $\sigma = 0.73$. This occurs because of the differences between the values of $\hat{K}$ for these two cases. Correspondingly, in the region with $t \sim 1$ there is a small difference between the periods.

In system of equations (11), the oscillations of all the functions are generated by vibrations of the transverse turbulent mass flow that are caused by oscillations of the function A_1; see Eq. (16). The function $\tilde{A}_1$ (we shall call it the amplitude function, since the oscillation amplitude depends on it) oscillates following the cosine law. After differentiation of Eq. (18) with respect to $\tilde{\tau}$, the corresponding differential equation of the second order is

$$\frac{\partial^2\tilde{A}_1}{\partial\tilde{\tau}^2} + \omega^2\tilde{A}_1 = -\omega^2\frac{f(t)}{t^2},$$

from which the dependence of $\tilde{A}_1$ on $\tilde{\tau}$ can be presented in the form

$$\tilde{A}_1 = -f(t)/t^2 + \tilde{A}_1'\cos(\omega\tilde{\tau} + \varphi_0),$$

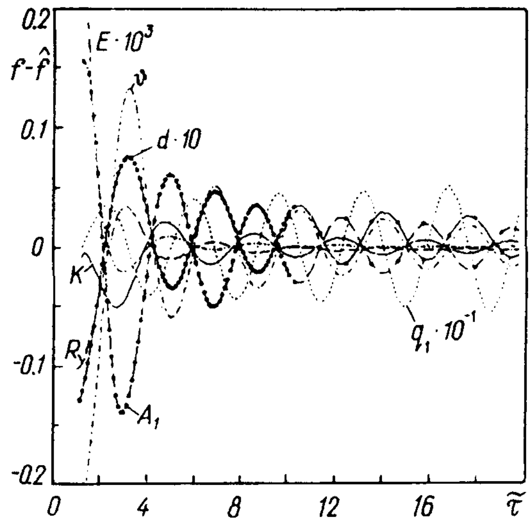

Fig. 4. Oscillating components of the functions of the model over the initial length.

where $\tilde{A}_1'$ is a function only of t. In contrast to the other functions with a tilde, $\tilde{A}_1$ has an additive component that is dependent on t. Since $\hat{A}_1 = 0$, this component represents a very small (at large values of t) mean value of the quantity A_1 about which its oscillations occur. The amplitude of the oscillations of $\tilde{q}'$ is associated with the amplitude of the oscillations of $\tilde{A}_1'$ by the relation

$$\tilde{q}' = \tilde{A}_1' t/\omega . \tag{26}$$

Representing $\tilde{q}$ in Eq. (13) in the form of the product $\tilde{q} = \tilde{q}'(t)\tilde{q}''(\tilde{\tau})$, we find the extent of the change of $\tilde{q}'(t)$ from the ordinary differential equation

$$\frac{t d\tilde{q}'}{\tilde{q}' dt} = p^{-1}\left[\frac{1}{\hat{R}}\left(2 - \frac{\hat{d}'}{3}\right) - 10\hat{d}'\hat{K} - \alpha_1 + p + 4\hat{q}\right] . \tag{27}$$

When $\tau \to \infty$, the right-hand side of Eq. (27) tends to a constant value. This determines the power-law character of the change in the amplitude of the oscillations of $\tilde{q}$ and, consequently, of all the remaining functions. This problem will be considered in detail in an analysis of the final stage of decay.

The next step in verifying the relations obtained is comparison of the oscillatory components in a numerical solution with their analytical expressions. For this purpose, from the numerical solution of the system of equations (5)-(7) we will subtract the solution of system (9), (10), and (15) for the corresponding functions. For a more accurate separation of oscillatory components one must select boundary conditions for the circumflexed functions that may differ somewhat from the boundary conditions of system (5)-(7). The oscillatory components plotted in Fig. 4 correspond to the following set of differences $\hat{f} - f$ in the initial section: $\hat{K} - K = 9.0\cdot10^{-3}$, $\hat{R} - R = 1.0\cdot10^{-1}$, $\hat{\vartheta} - \vartheta = 0.162$, $\hat{E} - E = -2.7\cdot10^{-4}$, $\hat{d} - d = 1.29\cdot10^{-2}$. The initial length of the solution trajectory was selected for comparison, since here the amplitudes are higher and oscillations are more noticeable, and in order to verify how well the far-region approximation works for moderate values of T_p.

We will analyze the mutual positions of the phases of the oscillations of the individual functions in Fig. 4. The oscillations of E and ϑ are in opposite phase, since these energies are partially transformed into one another; a decrease in one of them causes an increase in the other. A decrease in the pulsations of density in the first place causes an increase in the pulsations of the vertical velocity component; the oscillations of K are in phase with the oscillations of E and are more pronounced. There are two positions in which the pulsational turbulent transverse mass flow is equal to zero. We shall call them A and B. In position A the functions ϑ, R, and d are at the maximum, whereas E, K, and A_1 are minimal; the sign of the mass flow changes from plus (upward) to minus (downward); after surfacing of the light molecules, this situation corresponds to a greater density stratification than is indicated

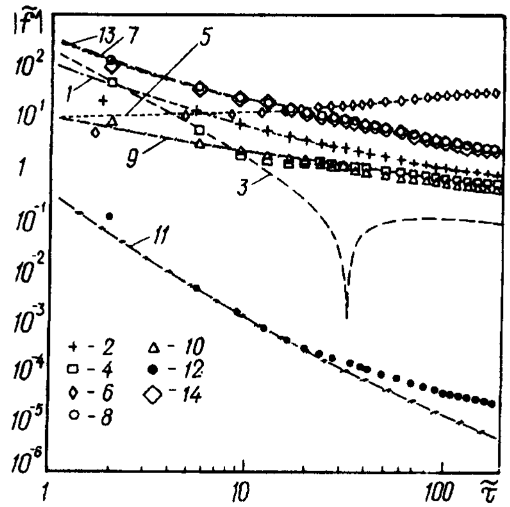

Fig. 5. Comparison of the amplitudes of the oscillations of the functions in the model (points) with those calculated by formulas (21)-(26) (curves): 1, 2) $|\tilde{K}'|$; 3, 4) $|\tilde{R}'|$; 5, 6) $|\tilde{q}'|$; 7, 8) $\tilde{\vartheta}'$; 9, 10) $|\tilde{E}'|$; 11, 12) $|\tilde{a}'|$; 13, 14) $|\tilde{A}_1'|$.

by the Fr and N_{BV} numbers. In this position velocity pulsations are partially quenched. In the opposite equilibrium position q (B), the turbulent mass flow changes its sign from minus to plus; after the lighter molecules descended downward, the density field becomes more uniform, density pulsations are quenched, and velocity pulsations increase.

Having distinguished oscillations, it is possible to analyze the change in their amplitudes. Figure 5 presents a comparison of the amplitudes calculated by solution of total system of equations (5)-(7) with the amplitudes calculated by solving the smoothed system of equations (9) and (15) together with Eqs. (27) and (21)-(26). Oscillatory components that are small compared to the mean value of the functions are distinguished with a some loss of accuracy. For example, in Fig. 5 the amplitudes of oscillations of $\tilde{E}$, which are especially small begin to deviate with time from the predicted values. For more exact discrimination of the components $\tilde{f}$ it is necessary to more accurately select the initial conditions for $\hat{f}$. The large amplitude deviation for $\tilde{R}$ near $\tilde{\tau} = 20$ has a fundamental character. At just this value of $\tilde{\tau}$ the change in the sign of the amplitude of $\tilde{R}'$ occurs. Near this point one should take into account the next terms of the expansion in the small parameter, and this should lead to the allowance for dispersion. By the way, in numerical computation the change in the period of oscillations of $\tilde{R}$ near this point is evident, but the period of oscillations common for all the remaining functions does not change. Evidently here we have inversion of the internal wave for $\tilde{R}$.

Discussion and Conclusions. A detailed analysis is made for a system of differential equations (1) that describe the evolution of homogeneous turbulence in the presence of a constant gradient of the density field.

The use of an approximate small-parameter method in conjunction with the method of many scales made it possible to isolate mathematical systems that describe regular oscillations (10) and variations averaged over these oscillations (9) and (15).

The use of the method of many scales is based here on the fact that turbulent characteristics are practically unchanged in the far region during a period of one wave. This difference in the time scales of the processes makes it possible to carry out their mathematical separation.

The change in the amplitude of the oscillations of q is described by differential equation (13), and the amplitudes of the oscillations of the remaining functions and their phases are related to the amplitude and phase of the oscillations of q by algebraic formulas relations (21)-(26). For the oscillatory component of the mass flow $\tilde{q}$ we obtained wave equation (20), the change of the frequency of oscillations in which occurs according to Eq. (19).

As a result of the analysis we can give the following qualitative description of the processes studied.

At the initial moment of time, due to intense mixing behind a grid or any other turbulizing device, turbulent molecules turn are swept into flow regions with a mean density that differs from the density of these fields.

Therefore, they experience the action of the resultant of the force of their weight and of the Archimedes torce. The resultant of these forces depends linearly on the shift of the molecule from the equilibrium position. On the other hand, the presence of a restoring force proportional to the displacement from equilibrium leads to the appearance of harmonic oscillations.

From the pulsations of velocity field under the action of the mean density gradient, regular oscillations are distinguished that constitute an internal gravity wave. The energy of this wave is drawn from the pulsation of the density field and from the velocity field pulsations. The presence of terms proportional to the flux $\hat{q}$ averaged over fluctuations, Eqs. (9), corresponds mathematically to the reverse process of the transition of the energy of regular oscillations into the energy of small-scale irregular pulsations. In the final stage of degeneration these terms turn out to be predominant in the equations.

The relation $\hat{A}_1 = 0$ is the principal one for the analysis in the entire far field, which is defined in the work as satisfying the condition $T_\rho >> 1$. However, it is possible to define it equivalently but differently as a region in which the relation $\hat{A}_1 = 0$ is satisfied approximately. From the physical point of view this expression specifies equilibrium state of the mutual values of the squared density pulsations and of the square of pulsations of the vertical velocity component.

In a turbulent gravity wave, a change in the energy of the density pulsations about the equilibrium position changes the energy of vertical velocity pulsations and vice versa. This results in attenuating oscillations of the quantity $A_1 = K + \vartheta(d(K - 1/3) - 2/3)$ with respect to the mean zero value. The oscillations of A_1 initiate oscillations with a $\pi/2$ phase shift of the turbulent transverse mass flow; this makes all the remaining functions in system (5) oscillate with the same frequency.

The approximate analytical solutions constructed in the work are confirmed by numerical calculation.

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NOTATION

τ^*, dimensional time; U, flow velocity; M, dimension of grid cell; $\tau = \tau^* U/M$, dimensionless time; $E^* = (\overline{u_1^2} + \overline{u_2^2} + u_3^2)$, kinetic energy of turbulence; $E = E^*/U^2$, dimensionless kinetic energy; $R_{22} = \overline{u_2^2}/U^2$, vertical component of the tensor of velocity pulsations; ε_ρ, rate of dissipation of density pulsations; ε_u, rate of dissipation of velocity pulsations; $T_u = (E^* U)/(\varepsilon_u M)$, time scale of velocity field; $T_\rho = (\overline{\rho^2} U)/(\varepsilon_\rho M)$, time scale of density field; $Q = (-\overline{u_2 \rho})/(UM d\overline{\rho}/dx_2)$, dimensionless turbulent transverse mass flow; $\Theta = \overline{\rho^2}/(M d\overline{\rho}/dx_2)^2$, square of density pulsations; σ, molecular Prandtl number; $\varepsilon = \mathrm{Fr}^2$, small parameter; $\mathrm{Fr} = N_{\mathrm{BV}} M/U$, Froude number; $N_{\mathrm{BV}} = (q d\overline{\rho}/\overline{\rho} dx_2)^{1/2}$, Brunt-Väisälä number; $R_\lambda = (5ET_u \mathrm{Re})^{1/2}$, turbulent Reynolds number; $\mathrm{Re} = UM/\nu$, Reynolds number.

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